Noncommutative Deformation Quantization and Gauge Interactions

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Solvay Workshop on
Higher Spin Gauge Theories, Topological Field Theory
and Deformation Quantization

[Brussels, 17 - 21 February 2020]

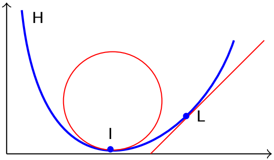
Outline

Higher Spin Interaction Problem

What is a mathematical structure underlying fundamental interactions?

LOW	INTERMEDIATE	HIGHER
Standard Model	(SU)GRA	HS theories
$0 \quad \frac{1}{2} \quad 1$	$\frac{3}{2} \quad 2$	$\frac{5}{2} \quad 3 \quad s$
Geometry of vector bundles with connexion	(super-) Riemannian geometry	???

The higher spin particles have no individual meaning upon switching on interaction.

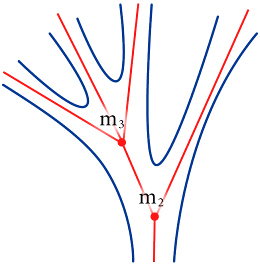


Strong Homotopy Algebras and Fundamental Interactions

Strong Homotopy Algebras
provide a universal control of gauge interactions whenever the EoM are brought into the form:

$$D\Phi = m_2(\Phi, \Phi) + m_3(\Phi, \Phi, \Phi) + \cdots$$

D squares to zero and differentiates m's.



String Field Theory:

- Φ is a string field (fermionic)
- D is a BRST operator associated to a conformal background
- m's are tree level string amplitudes

Higher Spin Gravity:

- Φ is a collection of differential forms on the space-time manifold
- $D = d$ is the exterior differential on forms
- m's are interaction vertices

Integrability condition

$$D^2 = 0 \quad \Leftrightarrow \quad \sum_{k+l=n} \pm m_k(\ldots, m_l(\ldots), \ldots) = 0\,, \quad n = 4, 5, \ldots,$$

defines the structure of a (minimal) A_∞-algebra constituted by

$$m_k(a_1, a_2, \ldots, a_k)\,, \qquad k = 2, 3, \ldots$$

If all m's are skew-symmetric, then we get a (minimal) L_∞-algebra.

$$A_\infty \quad \Leftrightarrow \quad \text{(open strings)}\,, \qquad L_\infty \quad \Leftrightarrow \quad \text{(closed strings)}$$

[E. Witten, B. Zwiebach, M. Gaberdiel, T. Erler, S. Konopka, I. Sachs, ...]

Strong Homotopy Algebras *per se*

Let $V = \bigoplus V_n$ denote the $\mathbb{Z}$-graded vector space of fields Φ.

The first two Stasheff's identities for m's read

- $$m_2(m_2(a,b),c) + (-1)^{|a|} m_2(a, m_2(b,c)) = 0\,, \quad \forall a,b,c \in V$$

  amounts to associativity of the product

  $$ab := (-1)^{|a|} m_2(a,b)\,.$$

  Let A denote the corresponding associative algebra.

- $$(\delta m_3)(a,b,c,d) := (-1)^{|a|} a m_3(b,c,d) + m_3(ab,c,d)$$

  $$+(-1)^{|a|} m_3(a,bc,d) + (-1)^{|a|+|b|} m_3(a,b,cd) + m_3(a,b,c)d = 0$$

  δ is the Hochschild differential, m_3 defines a cohomology class of $HH^3(A)$.

Gauge Interactions via Deformations of SHA

The associative algebra A is usually known

$$D\Phi = m_2(\Phi, \Phi) + m_3(\Phi, \Phi, \Phi) + m_4(\Phi, \Phi, \Phi, \Phi) + \cdots$$

Interaction Problem: Given m_2, find all higher interaction vertices obeying formal integrability.

Deformation interpretation: We are interested in deformations of an associative algebra A in the category of minimal A_∞-algebras:

$$A = (V; m_2, 0, 0, \dots) \quad \longrightarrow \quad A_\infty = (V; m_2, \lambda m_3, \lambda^2 m_4, \dots)\,,$$

λ being a formal deformation parameter (coupling constant).

For $4D$ HS gravity the problem was solved by M. A. Vasiliev in the late 1980's.

Inner Deformations of Families

Typically, associative algebras $A=(V,m_2)$ underlying 'free gauge theories' either involve some free parameters or can be included into n-parameter families

$$m_2 = m_2(t_1, t_2, \ldots, t_n)\,.$$

Theorem. *Any* 2*-parameter family of* A_∞*-structures*

$$m(t,s) = m_1 + m_2 + m_3 + \cdots$$

can be deformed into a 3*-parameter family* $m(\lambda,t,s)$ *satisfying the equations*

$$m'_\lambda(\ldots) = \sum \pm m(\ldots m'_t(\ldots)\ldots m'_s(\ldots)\ldots)\,,$$

$$m(0,t,s) = m(t,s)\,.$$

[E. Skvortsov & Sh, 2019]

Minimal Deformations of DG-algebras

Corollary. *Given a* 1*-parameter family of dg-algebras* $A=(V,\mu(t),\partial)$ *such that*

$$\partial : V^n \to V^{n-1}\,, \qquad \mu(t) : V^n\otimes V^m \to V^{n+m}\,,$$

one can define a minimal A_∞*-structure* $m=(\mu(t),\lambda m_3,\lambda^2 m_4,\ldots)$, *where*

$$\boxed{m_3(a,b,c)=\mu(\mu'_t(a,b),\partial c)\,, \qquad \forall a,b,c\in V}$$

If m_3 *represents a nonzero class of* $HH^3(A)$, *then the deformation is nontrivial.*

Indeed, $\quad m_2=\mu(t)\,, \quad m_1=s\partial\,, \qquad m(t,s)=m_1+m_2\,.$

A New Class of Integrable Models?

A free theory:

$$D\Phi = m_2(\Phi, \Phi)$$

- D is a differential, $D^2 = 0$.
- $m(t, s) = m_1 + m_2$ is a 2-parameter family of dg-algebras.

An interacting theory:

$$D\Phi = m_2(\Phi, \Phi) + \lambda m_3(\Phi, \Phi, \Phi)$$

$$+\lambda^2 m_4(\Phi, \Phi, \Phi, \Phi) + \cdots$$

- $m = m_2 + \lambda m_3 + \lambda^2 m_4 + \cdots$ is a solution to

$$m'_\lambda = \sum \pm m(..., m'_t, ..., m'_s, ...)$$

The integration flow:

$$\Phi'_\lambda = \sum \pm m(\Phi, \ldots, \Phi, \Phi'_t, \Phi, \ldots, \Phi, m'_s(\Phi, \ldots, \Phi), \Phi, \ldots, \Phi)\,,$$

$$\Phi|_{\lambda=0} = \Phi\,,$$

$$\Phi = \Phi + \lambda m_2(\Phi'_t, (m_1)'_s(\Phi)) + \cdots\,.$$

A New Class of Integrable Models?

Illustration

In the simplest situation $\quad \Phi = \varphi + \psi \in V = V_0 \oplus V_1\,, \quad V_0 \simeq V_1,$

$$\partial = \mathrm{id} : V_1 \to V_0\,, \qquad \partial^2 = 0\,.$$

The free equations $\quad D\psi = \psi * \psi\,, \qquad D\varphi = \varphi * \psi - \psi * \varphi$

admit a 'pure gauge' solution: $\quad \psi = g^{-1} * Dg\,, \quad \varphi = g^{-1} * \varphi_0 * g\,.$

$$\text{The solution space} = \{ \quad \varphi_0 \in V_0 \quad | \quad D\varphi_0 = 0 \quad \}\,.$$

Applying integration flow yields

$$\psi = \psi + \psi' * \varphi + \psi' * \varphi' * \varphi + (\psi' *' \varphi) * \varphi + \tfrac{1}{2}\psi'' * \varphi * \varphi + \cdots\,,$$

$$\varphi = \varphi + \varphi' * \varphi + \cdots\,, \qquad\qquad (\lambda = 1).$$

Other examples of integration flow:
[Seiberg & Witten, 1999; Prokushkin & Vasiliev, 1999]

Gerstenhaber's Deformation Theory

A formal deformation of an associative algebra A over k is the algebra $A[[t]]$ with a new $k[[t]]$-linear and associative product:

$$a * b = ab + t\phi_1(a,b) + t^2\phi_2(a,b) + \cdots\,, \qquad \forall a, b \in A\,.$$

Associativity requires:

- $(\delta\phi_1)(a,b,c) := a\phi_1(b,c) - \phi_1(ab,c) + \phi_1(a,bc) - \phi_1(a,b)c = 0$

  δ is the Hochschild differential, ϕ_1 defines an element of $HH^2(A)$

- $\delta\phi_2 = \frac{1}{2}[\phi_1, \phi_1]$, where

  $$[\phi_1,\phi_1](a,b,c) := \phi_1(\phi_1(a,b),c) - \phi_1(a,\phi_1(b,c))$$

  is the Gerstenhaber bracket. Since δ differentiates the GB, $[\phi_1,\phi_1]$ defines an element of $HH^3(A)$.

Noncommutative Deformation Quantization

- ☺ $HH^2(A)$ is isomorphic to the space of first-order deformations. $(\dim HH^2(A) =$ the number of coupling constants.)
- ☹ $HH^3(A)$ accommodates possible obstructions to integrability of first-order deformations.

The Gerstenhaber bracket passes through the cohomology making $HH^\bullet(A)$ into a graded Lie superalgebra.

Definition. *A noncommutative Poisson bracket on A is defined by an element $\Pi \in HH^2(A)$ such that $[\Pi, \Pi] = 0$.*

Problem: Given a noncommutative Poisson bracket $\Pi \in HH^2(A)$, define a formal deformation

$$a * b = ab + t\Pi(a, b) + O(t^2)\,.$$

Examples of Noncommutative Poisson Structures

- $A = C^\infty(M)$

  KHR theorem: $HH^2(A) \simeq$ (bivector fields on M)

  $[\Pi, \Pi] = 0 \quad \Leftrightarrow \quad$ Jacobi identity for the Poisson bivector

  Kontsevich Formality Theorem $\quad \Rightarrow \quad$ Any Poisson manifold admits a deformation quantization ($HH^3(A) \neq 0$).

- $A = C^\infty(M) \otimes \mathrm{Mat}_n(\mathbb{R})$

$$\Pi(a \otimes \alpha, b \otimes \beta) = \{a, b\} \otimes \alpha\beta$$

$\forall a, b \in C^\infty(M)$ and $\alpha, \beta \in \mathrm{Mat}_n(\mathbb{R})$.

Examples of Noncommutative Poisson Structures

- Noncommutative two-torus:

$$A_\theta \ni a = \sum_{n,m\in\mathbb{Z}} a_{nm} U^n V^m \,, \qquad UV = e^{2\pi i\theta} VU\,.$$

  Derivations:

$$\delta_u(U^nV^m) = 2\pi i n U^n V^m\,, \qquad \delta_v(U^nV^m) = 2\pi i m U^n V^m\,.$$

  Poisson bracket: $\qquad \Pi(a,b) = \delta_u(a)\delta_v(b)\,.$

- Skew group algebra $A = \mathbb{R}[x,y] \rtimes \mathbb{Z}_2$:

$$\mathbb{Z}_2 = \{1,\kappa\}\,, \quad \kappa^2 = 1\,, \quad \kappa f(x,y) = f(-x,-y)\kappa\,.$$

  In addition to the canonical Poisson bracket, we have

$$\Pi(a,b) = \frac{a(x,y)-a(-x,y)}{2x}\cdot\frac{b(-x,y)-b(-x,-y)}{2y}\kappa\,.$$

  Moreover, $a*b = ab + \nu\Pi(a,b)$ is the full deformation!

Deformations via Injective Resolutions

A model of an algebra A is given by a dg-algebra $(\mathcal{B}, d)$ together with an algebra homomorphism $\varepsilon : A \to \mathcal{B}_0$ such that the following sequence is exact:

$$0 \longrightarrow A \xrightarrow{\varepsilon} \mathcal{B}_0 \xrightarrow{d} \mathcal{B}_1 \xrightarrow{d} \mathcal{B}_2 \xrightarrow{d} \cdots$$

$$A \simeq \operatorname{Im} \varepsilon = \operatorname{Ker}(d : \mathcal{B}_0 \to \mathcal{B}_1) \subset \mathcal{B}_0 \,.$$

Define the dg-subalgebra $\mathcal{B}^A \subset \mathcal{B}$ of A-invariant elements:

$$\mathcal{B}^A = \{ b \in \mathcal{B} \quad | \quad [a, b] = 0 \,, \quad \forall a \in A \} \,.$$

If $\mathcal{B}$ is an injective A-bimodule, then

$$H^n(\mathcal{B}^A, d) \simeq HH^n(A) \,.$$

Theorem. *Any cohomology class $[\lambda] \in H^2(\mathcal{B}^A, d)$ with a representative $\lambda \in Z(\mathcal{B})$ defines an integrable deformation of A.*

[E. Skvortsov & A. Sh, 2018]

Example: Deformation of Skew Group Algebras

- Skew group algebra: $A = \mathbb{R}[x^1, x^2] \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2 = \{1, \kappa\}$ and

$$\kappa^2 = 1\,, \quad \kappa x^i = -x^i \kappa\,, \quad i = 1, 2\,.$$

- A model of A: $\quad \mathcal{B} = \mathcal{B}_0 \oplus \mathcal{B}_1 \oplus \mathcal{B}_2 = \mathbb{R}[x, p, dp] \rtimes \mathbb{Z}_2$,

$$\kappa p_i = -p_i \kappa\,, \quad \kappa dp_i = -dp_i \kappa \quad i = 1, 2\,,$$

$$\mathcal{B}_n \ni f = f^{i_1 \cdots i_n}(x, p, \kappa) dp_{i_1} \wedge \cdots \wedge dp_{i_n}\,, \qquad d = dp_i \wedge \frac{\partial}{\partial p_i}\,.$$

$$f \circ g = f e^{\frac{\overleftarrow{\partial}}{\partial p_i} \frac{\overrightarrow{\partial}}{\partial x^i}} g\,.$$

Clearly, $H(\mathcal{B}, d) \simeq A$ and $\varepsilon : A \to \mathcal{B}_0$ is a natural embedding.

- The centre $Z(\mathcal{B})$ is generated by dp_i and $\kappa\, e^{-2x^i p_i}$.

Example: Deformation of Skew Group Algebras

- The central cohomology in degree 2 is generated by the d-closed forms

$$dp_1 \wedge dp_2 \qquad \& \qquad \kappa\, e^{-2x^i p_i} dp_1 \wedge dp_2\,.$$

- They give a 2-parameter deformation $A_{\hbar,\nu}$ of the skew group algebra:

$$a * b = ab + \hbar\, \epsilon^{ij} \frac{\partial a}{\partial x^i} \frac{\partial b}{\partial x^j}$$

$$+\nu\kappa\epsilon^{ij} \int_{-1<t<s<1} dtds \left(\frac{\partial a}{\partial x^i}\right)(sx) \left(\frac{\partial b}{\partial x^j}\right)(tx) + \cdots\,.$$

$A_{1,\nu}$ is a symplectic reflection algebra:

$$[x^i, x^j] = \epsilon^{ij}(1+\nu\kappa)\,, \qquad \kappa x^i = -x^i\kappa\,, \qquad \kappa^2 = 1\,.$$

$A_{1,0} \otimes A_{1,0}$ is the (extended) HS algebra underlying 4D HS gravity [E. Fradkin & M. Vasiliev, 1986].

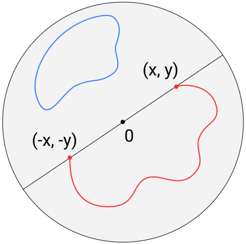
NDQ and Topological Strings

Let G be a finite group acting linearly on a vector space V.
There is a close relationship between the algebras

$$S(V^*) \rtimes G \qquad \& \qquad S(V^*)^G$$

$S(V^*)^G \simeq$ algebra of polynomial functions on V/G.

- Formality Theorem for orbifolds?
  [G. Halbout, J-M. Oudom & Xian Tang, 2011]
- The Poisson Sigma Model
  provides deformation quantization for $G = e$.
  [A. S. Cattaneo & G. Felder, 1999]
- Topological Quantum Mechanics on S^1
  reproduces the 1-st order deformation.
  [Si Li & Keyou Zeng, 2018]



$\mathbb{R}^2 / \mathbb{Z}_2$

Geometrization: the First Level

The RHS of the 'master equation'

$$D\Phi = m_2(\Phi,\Phi) + m_3(\Phi,\Phi,\Phi) + m_4(\Phi,\Phi,\Phi,\Phi) + \cdots$$

looks like Taylor's expansion in the vicinity of the vacuum solution $\Phi = 0$.

Let $\Phi = \varphi + \psi \in V_0 \oplus V_1 = V$ $\quad\Rightarrow\quad$ graded manifold M

Equations for component fields:

$$D\psi^a = f^a_{bc}(\varphi)\psi^b\psi^c\,, \qquad D\varphi^i = V^i_a(\varphi)\psi^a\,.$$

Formal integrability ($D^2 = 0$) $\quad\Leftrightarrow\quad$ Defining relations of a Lie algebroid:

$$f^d_{ab}f^e_{dc} + V_c f^e_{ab} + cycle(a,b,c) = 0\,, \qquad [V_a, V_b] = f^c_{ab}V_c\,.$$

$V_a = V^i_a(\varphi)\frac{\partial}{\partial\varphi^i}$ is the anchor of the Lie algebroid.

Geometrization: the First Level

$$V_a^i(\varphi) = V_{aj}^i \varphi^j + \cdots , \qquad f_{ab}^c(\varphi) = f_{ab}^c(0) + f_{abi}^c \varphi^i + \cdots$$

- $\varphi^i = 0$ is a phys. vacuum and a singular point of the LA, $V_a(0) = 0$.
- $f_{ab}^c(0)$ are the structure constants of an isotropy Lie algebra L
- $\{V_{aj}^i\}$ define a linear representation of L

$$D\psi^a = f_{bc}^a(0)\psi^b\psi^c + \cdots , \qquad D\varphi^i = V_{aj}^i \varphi^j \psi^a + \cdots .$$

Lie algebra: $[e_a, e_b] = f_{ab}^c(0) e_c, \quad [e_a, \theta^i] = V_{aj}^i \theta^j, \quad [\theta^i, \theta^j] = 0.$

Inclusion of an interaction amounts to the deformation of a Lie algebra structure in the category of Lie algebroids.

Geometrization: the Second Level

- The carrier manifold M of a Lie algebroid may have a non-trivial topology, which is invisible within perturbative approach.
- There may be other singular points (vacua) giving rise to other isotropy Lie algebras L (free theories).
- There is a generalization to n-Lie algebroids $\Phi \in V = \bigoplus_{l=0}^{n} V_l$.

- Much as a Lie algebra can be integrated to a Lie group, an n-Lie algebroid can be integrated to an n-Lie groupoid.

$$\begin{tikzcd}
\text{Lie algebra} \arrow[r, "int"] \arrow[d, "def"] & \text{Lie group} \arrow[d] \\
\text{Lie algebroid} \arrow[r, "int"] & \text{Lie groupoid?}
\end{tikzcd}$$

Question: What is a Lie groupoid integrating the HS Lie algebroid?

Antsimmetrization Map and Noncommutative Geometry

There is a natural relationship between A_∞- and L_∞-algebras:

$$l_n(a_1, a_2, \dots, a_n) = m_n(a_1 \wedge a_2 \wedge \cdots \wedge a_n)\,.$$

In dual picture, an L_∞-structure is described by a homological vector field Q on a graded manifold: $|Q| = 1$, $Q^2 = 0$.

Similarly, an A_∞-structure is given by a homological vector field on a formal noncommutative manifold.

$$\begin{tikzcd} A_\infty \arrow[r, "\text{antisym}"] \arrow[d, "\text{inner def}"'] \arrow[dr, phantom, "\text{DO NOT COMMUTE!}"] & L_\infty \arrow[d, "\text{inner def}"] \\ A_\infty \arrow[r, "\text{antisym}"'] & L_\infty \end{tikzcd}$$

Lesson: *When dealing with deformations of L_∞-algebras it is fruitful to regard them (whenever possible) as coming from A_∞-algebras.*

Summary

- A simple method is proposed for the deformation quantization of non-commutative Poisson structures.
- It is found that any 1-parameter family of dg-algebras naturally deforms into a minimal A_∞-algebra.
- Any multi-parameter family of associative algebras is shown to give rise to an integrable non-linear gauge theory; in particular, various HS gravity models fall into this class of theories.

Summary

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THANK YOU!